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Analysis and optimization of the motion of a body controlled by means of a movable internal mass☆

F.L. Chernous'ko

Moscow, Russia

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Abstract

The controlled horizontal motion of a body in the presence of dry friction forces is investigated. Control is accomplished by means of a movable mass that can move within the body in a bounded range. Some simple modes of periodic relative motions of the movable mass, under which the entire system moves as a whole, are investigated. Constraints are imposed on the relative displacement, velocity and acceleration of the movable mass. The optimum parameters of this relative motion, under which the maximum mean velocity of the body is reached, are determined.

1. The equations of motion

A body can move over a horizontal plane owing to oscillatory internal motions. This phenomenon is due to the presence of an external resistance, such as the dry friction forces between the body and the plane. A similar phenomenon can occur when a body moves in a resistant medium under various resistance laws.

This paper examines a simple mechanical model of this phenomenon. Rectilinear periodic motions of a mass within the main body, under which the entire system moves progressively as a whole along a horizontal straight line when there are dry friction forces between the body and the plane, are investigated. Unlike Ref. 1, in the present study the internal mass does not interact with the external medium. The mean velocity of the body is optimized with respect to parameters of the relative motion.

Consider rigid body of mass M, which can move along a horizontal straight line. Inside the body there is a movable mass m, which also moves horizontally (Fig. 1). For brevity, below we will refer to the body of mass M and the internal mass as "body M" and "mass m," respectively. An external dry friction force R acts on body M. Body M and mass m interact with one another. The interaction force applied to body M will be denoted by F. Then a force $-F$ acts on mass m. We will use x to denote the coordinate of body M, v to denote its velocity, ξ to denote the displacement of mass m relative to body M, and u and w to denote the velocity and acceleration of mass m relative to body M, respectively.

We will write the kinematic equations of the rectilinear motion of mass m relative to body M in the form

$$\dot{\xi} = u, \quad \dot{u} = w \tag{1.1}$$

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E-mail address: chern@ipmnet.ru.

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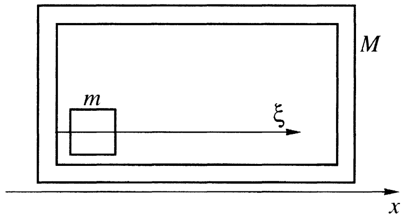

Fig. 1.

The dynamic equations of motion of body M and mass m along the horizontal straight line have the form

$$M\dot{\upsilon} = F + R, \quad m(\dot{\upsilon} + w) = -F \tag{1.2}$$

Eliminating the force F from Eq. (1.2), we obtain

$$\dot{\upsilon} = -\mu w + (M+m)^{-1}R, \quad \mu = m(M+m)^{-1} < 1 \tag{1.3}$$

By Coulomb's law, for dry friction we have

$$R = -(M+m)fg\,\mathrm{sign}\,\upsilon \quad \text{for} \quad \upsilon \neq 0; \quad |R| \leq (M+m)fg \quad \text{for} \quad \upsilon = 0 \tag{1.4}$$

Here g is the acceleration due to gravity and f is the constant coefficient of friction.

"Forward" and "backward" sliding of the body can occur for various values of the coefficient of friction. Therefore, it would be reasonable to consider the more general case of *anisotropic* dry friction, in which the resistance force is given by the relations

$$\begin{aligned} &R = -(M+m)f_+g, \quad \upsilon > 0 \\ &R = (M+m)f_-g, \quad \upsilon < 0 \\ &-(M+m)f_+g \leq R \leq (M+m)f_-g, \quad \upsilon = 0 \end{aligned} \tag{1.5}$$

In the general case of anisotropic dry friction (1.5), the equations of motion (1.3) of body M can be reduced to the form

$$\dot{\upsilon} = -\mu w - a_+ \quad \text{for} \quad \upsilon > 0, \quad \dot{\upsilon} = -\mu w + a_- \quad \text{for} \quad \upsilon < 0 \tag{1.6}$$

Here we have introduced the notation

$$a_+ = f_+g, \quad a_- = f_-g \tag{1.7}$$

A state of rest is maintained if the external force $(-\mu w)$ does not exceed the friction force, i.e.,

$$\upsilon = 0 \quad \text{for} \quad -a_+ \leq \mu w \leq a_- \tag{1.8}$$

Periodic laws of variation of the velocity of body M with period T and zero initial velocity are considered. Therefore, we have the conditions

$$\upsilon(0) = 0, \quad \upsilon(T) = 0 \tag{1.9}$$

The displacement of body M satisfies the equation and zero initial condition

$$\dot{x} = \upsilon, \quad x(0) = 0 \tag{1.10}$$

The mean velocity V of body M over the period T is given by the formula

$$V = \frac{x(T)}{T} \tag{1.11}$$

2. Relative motion

The motion of mass m relative to body M is described by Eq. (1.1) and occurs in the bounded interval of "allowed" relative displacements. Without loss of generality, we will specify the constraint on the displacement of mass m in the form

$$0 \le \xi(t) \le L \tag{2.1}$$

where $L > 0$ is a prescribed quantity. The constraint (2.1) can be determined, for example, by the dimensions of the cavity in body M or the existing stops that restrict displacement of mass m.

Let us consider periodic relative motions of mass m such that the functions $\xi(t)$, $u(t)$ and $w(t)$ are periodic functions with period $T > 0$. We will assume that the function $\xi(t)$ takes both extreme values that are permitted by inequalities (2.1). This assumption does not restrict the generality, since it can always be satisfied by introducing a shift along the ξ coordinate and a decrease in L. We choose the origin of the time t as the instant when $\xi = 0$. We then have

$$\xi(0) = 0, \quad \xi(T) = 0; \quad \xi(\theta) = L, \quad \theta \in [0, T] \tag{2.2}$$

Since the displacement $\xi(t)$ takes its minimum values at $t = 0$ and $t = T$, the velocity $u = \dot{\xi}$ satisfies the conditions

$$u(0) = 0, \quad u(T) = 0 \tag{2.3}$$

In addition, the inequalities

$$u(\varepsilon_1) > 0, \quad u(T - \varepsilon_2) < 0, \quad \varepsilon_1 > 0, \quad \varepsilon_2 > 0 \tag{2.4}$$

hold for fairly small ε_1 and ε_2.

In this paper we will confine ourselves to examining the two simple types of periodic relative motion of mass m that satisfy conditions (2.1)–(2.4). We will call them two-phase and three-phase motions. In *two-phase* motion the relative velocity $u(t)$ of mass m is piecewise constant, and there are two segments of constant velocity in a period of the motion. In *three-phase* motion the relative acceleration $w(t)$ of mass m is piecewise constant, and there are three segments of constant acceleration in a period of the motion.

It can easily be shown that the two-phase and three-phase motions considered have the smallest possible number of segments of constant velocity and acceleration, respectively, under the periodicity conditions imposed.

In fact, the displacement periodicity condition (2.2) cannot be ensured when there is only one segment where the velocity $u(t)$ is constant. If there are only two segments where of the acceleration $w(t)$ is constant, the velocity $u(t)$ will be a piecewise-linear, continuous function with two linear segments on the interval $[0,T]$. However, under conditions (2.3) such a function will have constant sign on the interval $[0, T]$. Therefore, condition (2.4) does not hold. Consequently there should be at least three segments of constant acceleration.

The two-phase and three-phase motions considered here are simple models of the following situations. Two-phase motion corresponds to the case of the greatest (theoretically unconstrained) possible relative acceleration and a constrained relative velocity. Three-phase motion corresponds to constrained relative acceleration. Thus, the two motions considered model different constraints on the relative motion of mass m, which correspond to different possibilities for the actuators that control this mass.

We present relations for the two-phase and three-phase motions that will be needed for the further discussion.

For two-phase motion we use τ_1 and τ_2 to denote the lengths of the segments of constant velocity, and we use u_1 and u_2 to denote the magnitudes of the velocity in these segments, respectively. We have

$$u(t) = u_1, \quad t \in (0, \tau_1); \quad u(t) = -u_2, \quad t \in (\tau_1, T), \quad T = \tau_1 + \tau_2 \tag{2.5}$$

Here u_1 and u_2 are positive constants, in accordance with the conditions (2.4). At the times $t = 0$ and $t = T$ we define the velocity $u(t)$ by relations (2.3). Then the relative acceleration $w(t)$ consists of velocity jumps at $t = 0$, $t = \tau_1$ and $t = T$, and it can be represented in the form

$$w(t) = u_1\delta(t) - (u_1 + u_2)\delta(t - \tau_1) + u_2\delta(t - T) \tag{2.6}$$

Here $\delta(t)$ is the Dirac delta function.

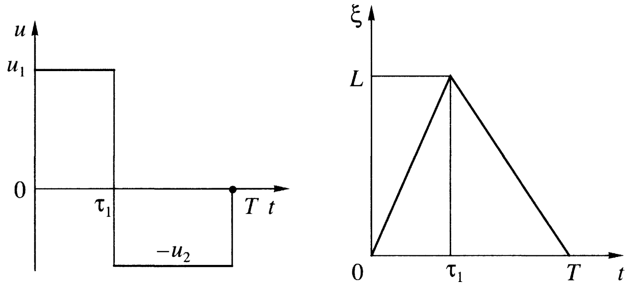

Fig. 2.

Of course, the velocity jumps at the beginning and end of a period can be combined into a single jump, but representation (2.6) is convenient because it agrees with conditions (2.3). The times $t=0$ and $t=T$ are regarded as equivalent.

Integrating relations (2.5) with the initial condition from (2.2), we obtain a piecewise-linear law for the displacement

$$\xi(t) = u_1 t, \quad t \in [0, \tau_1]; \quad \xi(t) = u_1\tau_1 - u_2(t-\tau_1), \quad t \in [\tau_1, T] \tag{2.7}$$

To satisfy conditions (2.2), it is necessary to require that

$$u_1\tau_1 = u_2\tau_2 = L, \quad \theta = \tau_1 \tag{2.8}$$

The two-phase motion is shown in Fig. 2. It can be characterized for given value of L by two independent positive parameters u_1 and u_2. The remaining parameters are expressed, according to conditions (2.8), in terms of u_1 and u_2 by the relations

$$\tau_1 = \theta = \frac{L}{u_1}, \quad \tau_2 = \frac{L}{u_2}, \quad T = L(u_1^{-1} + u_2^{-1}) \tag{2.9}$$

In the case of two-phase motion, an upper limit can be imposed on the relative velocity of mass m. We then have the constraints

$$0 < u_i \le U, \quad i = 1, 2 \tag{2.10}$$

where $U > 0$ is a specified constant.

For the three-phase motion we use τ_1, τ_2 and τ_3 to denote the lengths of the segments of constant relative acceleration, and we use w_1, $-w_2$ and w_3 to denote the acceleration values in these segments, respectively. We have

$$\begin{gathered} w(t) = w_1, \quad t \in (0, \tau_1); \quad w(t) = -w_2, \quad t \in (\tau_1, \tau_1+\tau_2) \\ w(t) = w_3, \quad t \in (\tau_1+\tau_2, T), \quad T = \tau_1+\tau_2+\tau_3 \end{gathered} \tag{2.11}$$

From the equality $\dot{u} = w$ and conditions (2.3) and (2.4), it follows that $w_1 > 0$ and $w_3 > 0$ in equalities (2.11). If $w_2 \le 0$, then taking the inequalities $w_1 > 0$ and $w_3 > 0$ into account, we conclude that the function $u(t)$ strictly increases for $t \in (0, \tau_1)$ and $t \in (\tau_1+\tau_2, T)$ and does not decrease for $t \in (\tau_1, \tau_1+\tau_2)$, contrary to the second condition in (2.3). Therefore, $w_2 > 0$.

We integrate Eq. (1.1) with the acceleration (2.11) and the initial conditions (2.3) and (2.2). We obtain the relative velocity of mass m in the three-phase motion

$$\begin{gathered} u(t) = w_1 t; \quad t \in [0, \tau_1]; \quad u(t) = w_1\tau_1 - w_2(t-\tau_1), \quad t \in [\tau_1, \tau_1+\tau_2] \\ u(t) = w_1\tau_1 - w_2\tau_2 + w_3(t-\tau_1-\tau_2), \quad t \in [\tau_1+\tau_2, T] \end{gathered} \tag{2.12}$$

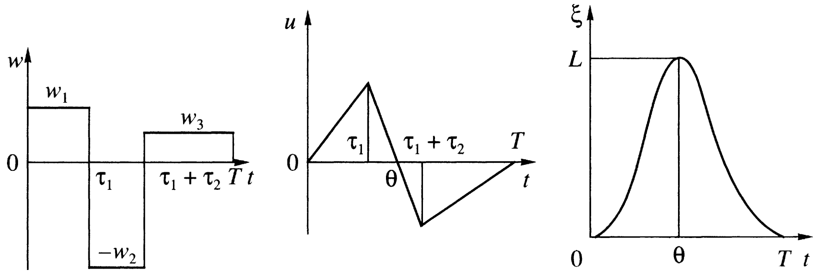

Fig. 3.

and its relative displacement

$$\xi(t) = \frac{w_1 t^2}{2}, \quad t \in [0, \tau_1]$$

$$\xi(t) = \frac{w_1\tau_1^2}{2} + w_1\tau_1(t-\tau_1) - \frac{w_2(t-\tau_1)^2}{2}, \quad t \in [\tau_1, \tau_1+\tau_2]$$

$$\xi(t) = \frac{w_1\tau_1^2}{2} + w_1\tau_1\tau_2 - \frac{w_2\tau_2^2}{2} + (w_1\tau_1 - w_2\tau_2)(t-\tau_1-\tau_2) + \frac{w_3(t-\tau_1-\tau_2)^2}{2}, \quad t \in [\tau_1+\tau_2, T] \tag{2.13}$$

From the second condition in (2.3) and relations (2.12) it follows that

$$w_1\tau_1 - w_2\tau_2 + w_3\tau_3 = 0 \tag{2.14}$$

Substituting the displacement (2.13) into the second condition of (2.2), we obtain

$$\frac{w_1\tau_1^2}{2} + w_1\tau_1\tau_2 - \frac{w_2\tau_2^2}{2} + w_1\tau_1\tau_3 - w_2\tau_2\tau_3 + \frac{w_3\tau_3^2}{2} = 0 \tag{2.15}$$

The maximum displacement $\xi(\theta) = L$ is reached in the interval $(\tau_1, \tau_1+\tau_2)$ when $u(\theta) = 0$. From (2.12) we find

$$\theta = \tau_1 + w_1\tau_1/w_2, \quad \theta \in (\tau_1, \tau_1+\tau_2) \tag{2.16}$$

We substitute expression (2.16) into (2.13) and write the last condition of (2.2). After simplification we obtain

$$w_1(w_1+w_2)\tau_1^2 = 2w_2L \tag{2.17}$$

The three-phase motion is illustrated in Fig. 3. This motion can be characterized for a given value of L by the three positive parameters w_1, w_2 and w_3. The parameters τ_1, τ_2 and τ_3 can be expressed using Eqs. (2.14), (2.15) and (2.17). From equality (2.14) we have

$$\tau_2 = (w_1\tau_1 + w_3\tau_3)/w_2 \tag{2.18}$$

Substituting expression (2.18) into equality (2.15), after some reduction we obtain

$$w_1(w_1+w_2)\tau_1^2 = w_3(w_2+w_3)\tau_3^2 \tag{2.19}$$

Solving (2.17)–(2.19) for τ_1, τ_2 and τ_3, we successively find

$$\tau_1 = \left[\frac{2w_2L}{w_1(w_1+w_2)}\right]^{\frac{1}{2}}, \ \tau_2 = \left(\frac{2L}{w_2}\right)^{\frac{1}{2}}\left[\left(\frac{w_1}{w_1+w_2}\right)^{\frac{1}{2}} + \left(\frac{w_3}{w_2+w_3}\right)^{\frac{1}{2}}\right], \ \tau_3 = \left[\frac{2w_2L}{w_3(w_2+w_3)}\right]^{\frac{1}{2}} \tag{2.20}$$

Using equalities (2.20), we express the period length $T = \tau_1 + \tau_2 + \tau_3$ in terms of the parameters w_1, w_2 and w_3. We obtain

$$T = \left(\frac{2L}{w_2}\right)^{\frac{1}{2}}\left[\left(\frac{w_1 + w_2}{w_1}\right)^{\frac{1}{2}} + \left(\frac{w_2 + w_3}{w_3}\right)^{\frac{1}{2}}\right] \tag{2.21}$$

An upper limit can be imposed on the acceleration of mass m in the three-phase motion. Then we have the following constraints for the parameters w_i

$$0 < w_i \le W, \quad i = 1, 2, 3 \tag{2.22}$$

where $W > 0$ is an assigned positive constant.

In Sections 3–6 we investigate and optimize the motion of body M with two-phase and three-phase relative motions of mass m in the general case of anisotropic dry friction (1.5). The maximum mean velocity V of body M (under the constraints (2.10) and (2.22)) is found from (1.11) for these motions.

3. Two-phase motion

We substitute the function $w(t)$ for the two-phase motion (2.6) into Eq. (1.6). From Eq. (1.6) and the conditions (1.9) it follows that the velocity $v(t)$ has three discontinuities in the interval $[0,T]$: two jumps at the endpoints, at which $v(t)$ decreases, and one large-amplitude jump within the interval, at which $v(t)$ increases. On the segments between the discontinuities, the absolute value of the velocity $v(t)$ decreases linearly, and the presence of a rest interval is also possible.

As a simple analysis shows, the two modes of motion of body M shown in Fig. 4a and b are possible. In mode a there is no rest interval preceding the discontinuity when $t \in (0, \tau_1]$, and in mode b there is a rest interval (t_1, τ_1). In Fig. 4c we show the case intermediate between modes a and b.

We will calculate the velocity $v(t)$ for both modes using (1.6), (2.6) and (1.9).

For mode a we obtain

$$\begin{aligned} v(t) &= -\mu u_1 + a_- t \quad \text{for} \quad t \in (0, \tau_1) \\ v(t) &= \mu u_2 + a_-\tau_1 - a_+(t - \tau_1) \quad \text{for} \quad t \in (\tau_1, T) \end{aligned} \tag{3.1}$$

The following condition must hold (see Fig. 4a)

$$v(\tau_1 - 0) = -\mu u_1 + a_-\tau_1 \le 0 \tag{3.2}$$

When relations (1.6) and (2.6) are taken into account, the second equality in (1.9) can be reduced to the form

$$v(T) = v(T - 0) - \mu u_2 = a_-\tau_1 - a_+\tau_2 = 0 \tag{3.3}$$

For mode b we have

$$\begin{aligned} v(t) &= -\mu u_1 + a_- t, \quad t \in (0, t_1) \\ v(t) &= 0, \quad t \in (t_1, \tau_1) \\ v(t) &= \mu(u_1 + u_2) - a_+(t - \tau_1), \quad t \in (\tau_1, T) \end{aligned} \tag{3.4}$$

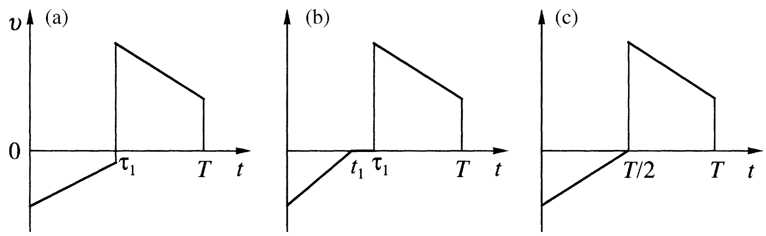

Fig. 4.

Here $t_1 \in (0, \tau_1)$ is the instant of a stop determined from the condition $v(t_1) = 0$. From relations (3.4) we obtain

$$t_1 = \mu u_1/a_- < \tau_1 \tag{3.5}$$

The second condition in (1.9) for mode b gives

$$v(T) = v(T-0) - \mu u_2 = \mu u_1 - a_+\tau_2 = 0 \tag{3.6}$$

Note that inequalities (3.2) and (3.5) are mutually exclusive and define modes a and b, respectively. We substitute relations (2.9) into these inequalities, as well as into formulae (3.3) and (3.6). Expressing u_2 in terms of u_1, we obtain

$$u_2 = a_+u_1/a_-, \quad u_1 \geq (La_-/\mu)^{\frac{1}{2}} \text{ for regime a} \tag{3.7}$$

$$u_2 = a_+L/(\mu u_1), \quad u_1 < (La_-/\mu)^{\frac{1}{2}} \quad \text{for regime b} \tag{3.8}$$

We calculate the total displacement $x(T)$ of body M over a period. For this purpose, we integrate $v(t)$ over the interval $[0,T]$ under the condition $x(0)=0$ (see (1.10)).

For mode a, after some simplifications that rely on inequalities (3.3), we obtain

$$x(T) = \frac{a_-\tau_1(\tau_1+\tau_2)}{2} \tag{3.9}$$

The mean velocity V of body M is defined by (1.11). For mode a, based on (3.9) and (2.9) we find

$$V = \frac{a_-\tau_1}{2} = \frac{La_-}{(2u_1)} \tag{3.10}$$

For mode b, using (3.4)–(3.6), we obtain

$$x(T) = \mu^2 u_1 \frac{\left[\left(\frac{u_1}{a_+}\right) + 2\left(\frac{u_2}{a_+}\right) - \left(\frac{u_1}{a_-}\right)\right]}{2} \tag{3.11}$$

We calculate the mean velocity (1.11) using (3.11) and (2.9). Eliminating u_2 with the aid of (3.8), for mode b we have

$$V = \frac{\left[\mu L + \left(\frac{\mu^2 u_1^2}{2}\right)(a_+^{-1} - a_-^{-1})\right]u_1}{L + \mu u_1^2 a_+^{-1}} \tag{3.12}$$

It is convenient to continue the analysis in dimensionless variables. We introduce the notation

$$u_0 = \left(\frac{La_-}{\mu}\right)^{\frac{1}{2}}, \quad c = \frac{a_+}{a_-} \tag{3.13}$$

and change to dimensionless parameters using the formulae

$$u_i = u_0 x_i, \quad i = 1, 2, \quad V = \frac{(\mu La_-)^{\frac{1}{2}}\Phi}{2} \tag{3.14}$$

Then relations (3.7), (3.8), (3.10) and (3.12) take the form

$$x_2 = cx_1, \quad x_1 \geq 1, \quad \Phi(x_1) = x_1^{-1} \text{ for regime a} \tag{3.15}$$

$$x_2 = \frac{c}{x_1}, \quad x_1 < 1, \quad \Phi(x_1) = [2c + x_1^2(1-c)]x_1(c+x_1^2)^{-1} \quad \text{for regime b} \tag{3.16}$$

4. Optimization of the two-phase motion

Let us find the values of the velocities u_1 and u_2 at which body M reaches the highest mean velocity V. This problem reduces to finding the value of the dimensionless parameter $x_1 = x_1^* > 0$ that provides the maximum Φ^* of the function $\Phi(x_1)$, which is defined by (3.15) and (3.16). We have

$$\Phi^*(c) = \max_{x_1}\Phi(x_1) = \Phi(x_1^*) \tag{4.1}$$

4.1. The case without constraints (2.10)

We will first assume that no upper limits are imposed on the velocities u_1 and u_2 or, consequently, on the parameters x_1 and x_2. We will find the maximum (4.1) when $x_1 \in (0, \infty)$.

According to relations (3.15), the function $\Phi(x_1)$ decreases monotonically when $x_1 \geq 1$. Let us investigate this function for extremum when $x_1 < 1$. For this purpose we will calculate its derivative using equality (3.16). We obtain

$$\frac{d\Phi}{dx_1} = \Phi_1(z)(c+z)^{-2} \tag{4.2}$$

Here we have introduced the notation

$$\Phi_1(z) = (1-c)z^2 + c(1-3c)z + 2c^2, \quad z = x_1^2 \tag{4.3}$$

An investigation of the quadratic trinomial $\Phi_1(z)$ (4.3) gives the following results. The trinomial $\Phi_1(z)$ has the real roots z_1 and z_2 when $c \geq 7/9$. If $c \in [7/9, 1]$, both roots lie in the range $z \geq 1$. Consequently, for $c \leq 1$ the function $\Phi_1(z)$ is positive in the interval $z \in (0, 1)$. Therefore, by virtue of (4.2), the function $\Phi(x_1)$ increases monotonically when $x_1 \in (0, 1)$. Consequently, for $c \leq 1$ the absolute maximum of the function $\Phi(x_1)$ is reached when $x_1 = 1$.

When $c > 1$, one of the roots of the quadratic trinomial (4.3), determined by

$$z_1(c) = c(c-1)^{-1}\frac{\left[1-3c+(9c^2+2c-7)^{\frac{1}{2}}\right]}{2} \tag{4.4}$$

lies in the interval (0, 1) and corresponds to the maximum of the function $\Phi(x_1)$.

Thus, the required maximum (4.1) and the corresponding value $x_1 = x_1^*$, which provides this maximum, are specified by the relations

$$\Phi^*(c) = 1, \quad x_1^* = 1 \quad \text{for} \quad c \leq 1 \tag{4.5}$$

$$\Phi^*(c) = \Phi(x_1^*), \quad x_1^* = z_1^{\frac{1}{2}}(c) \quad \text{for} \quad c > 1 \tag{4.6}$$

The quantity Φx_1^* is defined by formula (3.16). According to formulae (4.5) and (4.6), the maximum dimensionless mean velocity $\Phi^*(c)$ is reached when $c \leq 1$ in mode a and when $c > 1$ in mode b. Fig. 5 shows plots of $x_1^*(c) = z_1^{1/2}(c)$ and $\Phi^*(c)$ obtained using formulae (4.4)–(4.6).

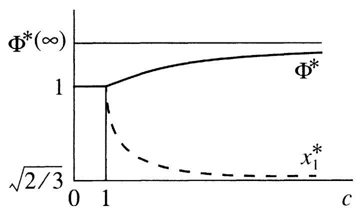

Fig. 5.

We note several properties of these plots. The function $x_1^*(c)$ decreases monotonically from $x_1^* = 1$ to $x_1^* = \left(\frac{2}{3}\right)^{1/2} \approx 0.816$ as c varies from 1 to ∞. The function $\Phi(c)$ increases monotonically from 1 to the value $\Phi*(\infty) = \left(\frac{4}{3}\right)\left(\frac{2}{3}\right)^{1/2} \approx 1.0887$ as c varies from 1 to ∞. Since the function $\Phi^*(c)$ does not exceed the value of $\Phi^*(\infty)$ for all $c>1$, the velocity advantage of optimum mode b (4.6) over mode a amounts to no more than 9%.

Let us return to the dimensional variables u_1, u_2 and τ_1, τ_2 using formulae (3.14) and (2.9). Introducing the additional notation (see (3.13))

$$\tau_0 = \frac{L}{u_0} = \left(\frac{\mu L}{a_-}\right)^{\frac{1}{2}}, \quad u_0 = \left(\frac{La_-}{\mu}\right)^{\frac{1}{2}} \tag{4.7}$$

according to (3.14), (3.15) and (4.5), for the optimum motion for $c \le 1$ we obtain

$$u_1 = u_0, \quad u_2 = cu_0, \quad \tau_1 = \tau_0, \quad \tau_2 = c^{-1}\tau_0, \quad V = \frac{(\mu La_-)^2}{2} \quad \text{for} \quad c \le 1 \tag{4.8}$$

For $c > 1$, using formulae (3.5), (3.14), (3.16), (4.6) and (4.7), we have

$$\begin{aligned} &u_1 = u_0 x_1^*(c), \quad u_2 = u_0 c/x_1^*(c), \quad \tau_1 = \tau_0/x_1^*(c), \\ &\tau_2 = \tau_0 c^{-1} x_1^*(c), \quad t_1 = \tau_0 x_1^*(c), \quad V = \frac{(\mu La_-)^{\frac{1}{2}}\Phi^*}{2} \quad \text{for} \quad c > 1 \end{aligned} \tag{4.9}$$

By virtue of formulae (4.9) and the properties of the function $x_1^*(c)$ noted above (see Fig. 5), the following relations hold for the interval lengths:

$$\begin{aligned} &\left(\frac{2}{3}\right)^{\frac{1}{2}}\tau_1 \le t_1 = [x_1^*(c)]^2\tau_1 < \tau_1, \\ &\frac{\tau_2}{\tau_1} = \frac{[x_1^*(c)]^2}{c} < [x_1^*(c)]^2 < 1 \quad \text{for} \quad c > 1 \end{aligned}$$

In the simplest case of isotropic dry friction with $a_+ = a_- = a$, we have $c = 1$. In this case, according to relations (4.7) and (4.8), the optimum mode is the mode for which

$$\begin{aligned} &u_1 = u_2 = u_0 = (La/\mu)^{\frac{1}{2}}, \quad \tau_1 = \tau_2 = \tau_0 = \left(\frac{\mu L}{a}\right)^{\frac{1}{2}}, \\ &T = 2\tau_0, \quad V = \frac{(\mu La)^{\frac{1}{2}}}{2} \quad \text{for} \quad c = 1 \end{aligned} \tag{4.10}$$

This case is shown in Fig. 4c.

Note that even when there are no upper bounds (2.10) for the relative velocities u_1 and u_2, the maximum mean velocity V of body M was found to be constrained according to equalities (4.8) and (4.9). For all values of the parameter c, it turned out to be confined to the fairly narrow range

$$(\mu La_-)^{\frac{1}{2}} \le 2V < \Phi^*(\infty)(\mu La_-)^{\frac{1}{2}}, \quad \Phi^*(\infty) < 1.09 \tag{4.11}$$

4.2. The case with constraints (2.10)

Let us now assume that the upper limit (2.10) is imposed on the relative velocity of point m. Using the notation (3.13) and (3.14), we change to dimensionless variables and represent the constraints (2.10) in the form

$$0 < x_i \le X, \quad i = 1, 2, \quad X = U/u_0 \tag{4.12}$$

The problem of finding the maximum mean velocity of body M reduces to maximizing (see (4.1)) the function $\Phi(x_1)$, specified by equalities (3.15) and (3.16) under constraints (4.12).

We will consider four cases in succession.

1) $c \le 1, x_1 \ge 1$. In this case, constraints (3.15) and (4.12) are compatible when $X \ge 1$ and specify the following range of variation of x_1: $x_1 \in [1, X]$.
2) $c > 1, x_1 \ge 1$. Constraints (3.15) and (4.12) are compatible when $X \ge c$ and specify the interval $x_1 \in \left[1, \frac{X}{c}\right]$.
3) $c \le 1, x_1 < 1$. Constraints (3.16) and (4.12) reduce to the inequalities $x_1 < 1$ and $c/X \le x_1 \le X$, which are compatible when $X > c$ and $X \ge c^{1/2}$. As a result, this case occurs when $X \ge c^{1/2}$, and it specifies the interval $x_1 \in \left[\frac{c}{X}, X\right]$ for $X < 1$ and the interval $x_1 \in \left[\frac{c}{X}, 1\right]$ for $X \ge 1$.
4) $c > 1, x_1 < 1$. Constraints (3.16) and (4.12) are compatible when $X \ge c$ and specify the interval $x_1 \in \left[\frac{c}{X}, 1\right]$.

In cases 1 and 2 the function $\Phi(x_1)$ (3.15) decreases monotonically when $x_1 \ge 1$ and reaches its maximum at the left-hand endpoint of the interval of possible variation of x_1, i.e., at $x_1 = 1$.

In case 3, as was shown above, the function $\Phi(x_1)$ (3.16) increases monotonically in the interval $x_1 \in (0, 1)$. Therefore, the required maximum is reached when $x_1 = X$ if $X < 1$ and when $x_1 = 1 - 0$ if $X \ge 1$.

In case 4 single maximum of the function $\Phi(x_1)$ (3.16) in the interval $x_1 \in (0, 1)$ is reached when $x_1 = z_1^{1/2}$, and the value of $z_1(c)$ is given by formula (4.4). Thus, the required maximum is reached here when $x_1 = z_1^{1/2}$ if $c/X \le z_1^{1/2}$ and when $x_1 = \frac{c}{X}$ if $\frac{c}{X} > z_1^{1/2}$.

Combining the results for cases 1 and 3, we find that for $c \le 1$ the required maximum is reached when

$$\begin{aligned} &1)\ x_1^* = X, \quad \Phi^*(c) = \Phi(X) \quad \text{for} \quad c^{\frac{1}{2}} \le X < 1 \\ &2)\ x_1^* = 1, \quad \Phi^*(x) = 1 \quad \text{for} \quad c \le 1,\ X \ge 1 \end{aligned} \tag{4.13}$$

Comparing the results for cases 2 and 4, we obtain the following results for $c > 1$:

$$\begin{aligned} &3)\ x_1^* = \frac{c}{X}, \quad \Phi^*(c) = \Phi\left(\frac{c}{X}\right) \quad \text{for} \quad 1 < c \le X < cz_1^{-\frac{1}{2}}(c) \\ &4)\ x_1^* = z_1^{\frac{1}{2}}(c), \quad \Phi^*(c) = \Phi\left(z_1^{\frac{1}{2}}(c)\right) \quad \text{for} \quad c > 1, \quad X \ge cz_1^{-\frac{1}{2}}(c) \end{aligned} \tag{4.14}$$

In equalities (4.13) and (4.14), the function Φ is defined by (3.15) and (3.16).

Thus, under constraints (4.12) the required maximum of the dimensionless velocity $\Phi^*(c)$ (4.1) is completely specified by (4.13) when $c \le 1$ and by (4.14) when $c > 1$.

Using relations (3.13), (3.14) and (2.9), we can return to the original dimensional variables.

Fig. 6 shows the regions corresponding to different cases of formulae (4.13) and (4.14) in the c, X plane. Under constraints (2.10) or (4.12) the two-phase motions considered can only occur for $X \ge c^{1/2}$ if $c \le 1$ and only when $X > c$ if $c > 1$, i.e., under the condition

$$X \ge \max\left(c^{\frac{1}{2}}, c\right) \tag{4.15}$$

The graphs of $X = c^{1/2}$ and $X = cz_1^{-1/2}(c)$ are indicated by the letters K and N, respectively, in Fig. 6. Together with the straight lines $c = 1$, $X = 1$ and $X = c$, they divide the region (4.15) into four regions, which are indicated by the

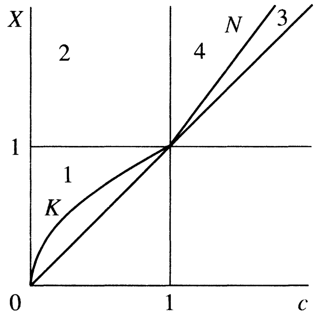

Fig. 6.

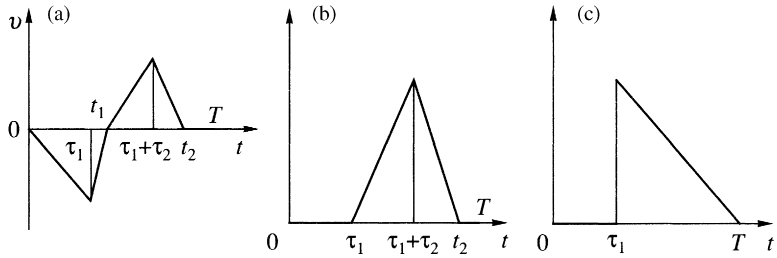

Fig. 7.

numbers 1, 2, 3 and 4 and correspond in succession to the four possibilities for solutions in formulae (4.13) and (4.14). In Fig. 6, motion mode *a* occurs in region 2, and mode *b* occurs in the remaining regions 1, 3 and 4.

We will separately examine the case of isotropic dry friction ($a_+ = a_- = a$) under constraints (2.10) or (4.12). For this purpose, we can return to the general formulae (4.13) or utilize the solution in dimensional variables for this case (4.10). We find that if the limiting permissible velocity is sufficiently high, i.e., if $X \geq 1$ or, in dimensional variables, $U \geq (\mu La)^{1/2}$, the motion (4.10) is optimum. If $X < 1$ (or $U < (\mu La)^{1/2}$), the two-phase motion considered cannot occur.

5. The three-phase motion

The motion considered is described by relations (1.6)–(1.9), into which the variation law (2.11) of the acceleration must be substituted. Taking into account the signs of the accelerations and the conditions imposed, we reach the conclusion that two modes, namely, mode *a* and mode *b*, which are shown in Fig. 7, are possible.

In mode *a*, the velocity $v(t)$ becomes negative when $t > 0$ and decreases up to the instant $t = \tau_1$, at which the acceleration changes in sign. Then, the velocity increases, passes through zero at a certain time $t = t_1 \in (\tau_1, \tau_1 + \tau_2)$ and continues to increase until the time $t = \tau_1 + \tau_2$. Then, $v(t)$ decreases and vanishes at $t = t_2 \in (\tau_1 + \tau_2, T)$. In this mode, body *M* moves "backward," i.e., $v(t) < 0$, at the beginning of a period.

In mode *b* body *M* is stationary when $t \in (0, \tau_1)$ and $t \in (t_2, T)$, where $t_2 \in (\tau_1 + \tau_2, T)$, and moves forward when $t \in (\tau_1, t_2)$. The velocity $v(t)$ is non-negative, increases in the interval $(\tau_1, \tau_1 + \tau_2)$ and decreases when $t \in (\tau_1 + \tau_2, t_2)$.

Integrating Eq. (1.6) for $w(t)$, which is given by relations (2.11), under the initial condition (1.9), for mode *a* we obtain

$$\begin{aligned}
v(t) &= (a_- - \mu w_1)t, \quad t \in [0, \tau_1] \\
v(t) &= (a_- - \mu w_1)\tau_1 + (\mu w_2 + a_-)(t - \tau_1), \quad t \in [\tau_1, t_1] \\
v(t) &= (\mu w_2 - a_+)(t - t_1), \quad t \in [t_1, \tau_1 + \tau_2] \\
v(t) &= (\mu w_2 - a_+)(\tau_1 + \tau_2 - t_1) - (\mu w_3 + a_+)(t - \tau_1 - \tau_2), \quad t \in [\tau_1 + \tau_2, t_2] \\
v(t) &= 0, \quad t \in [t_2, T]
\end{aligned} \tag{5.1}$$

We will present the conditions that must be satisfied for mode a. Since, according to relations (5.1), in the initial segment $v(t) < 0$, we have

$$\mu w_1 > a_- \tag{5.2}$$

The time t_1 is determined from the condition $v(t_1) = 0$. Therefore, we obtain

$$t_1 = \tau_1 + \frac{\mu w_1 - a_-}{\mu w_2 + a_-}\tau_1 = \frac{\mu(w_1 + w_2)}{\mu w_2 + a_-}\tau_1 \tag{5.3}$$

In order for body M to move forward, the time t_1 must occur before the interval $(\tau_1, \tau_1 + \tau_2)$ ends, i.e., $t_1 < \tau_1 + \tau_2$. Using condition (5.3), we can reduce this inequality to the form

$$\mu w_1 \tau_1 < \mu w_2 \tau_2 + a_-(\tau_1 + \tau_2) \tag{5.4}$$

Since the velocity is positive when $t \in (t_1, \tau_1 + \tau_2)$, the following inequality should hold:

$$\mu w_2 > a_+ \tag{5.5}$$

We determine the time t_2 from the condition $v(t_2) = 0$. According to relations (5.1), we obtain

$$t_2 = \tau_1 + \tau_2 + \frac{(\mu w_2 - a_+)(\tau_1 + \tau_2 - t_1)}{\mu w_3 + a_+} \tag{5.6}$$

This equation and the condition $t_2 \le T = \tau_1 + \tau_2 + \tau_3$ lead to the inequality

$$(\mu w_2 - a_+)(\tau_1 + \tau_2 - t_1) \le (\mu w_3 + a_+)\tau_3 \tag{5.7}$$

If the strict inequality holds in (5.7), i.e., when $t_2 < T$, realization of the state of rest $v = 0$ when $t \in [t_2, T]$ in mode a (5.1) requires that

$$\mu w_3 \le a_- \tag{5.8}$$

In the case of the equality sign in (5.7), i.e., when $t_2 = T$, condition (5.8) is not imposed.

Relations (5.2)–(5.8) together with the accompanying explanations define the conditions for the existence of mode a.

Let us calculate the total displacement $x(T)$ of body M for the period T. For this purpose, we integrate the velocity $v(t)$ (5.1) over the period. To do this, it is sufficient to sum the areas of the triangles under the graph of $v(t)$ in Fig. 7a taking into account the signs of these areas. We have the equality

$$x(T) = \frac{v(\tau_1)t_1}{2} + \frac{v(\tau_1 + \tau_2)(t_2 - t_1)}{2}$$

Substituting the appropriate expressions from (5.1), (5.3) and (5.6) into this equality, we obtain

$$\begin{aligned} x(T) &= \frac{\mu(a_- - \mu w_1)(w_1 + w_2)\tau_1^2}{2(\mu w_2 + a_-)} + \frac{(\mu w_2 - a_+)(\tau_1 + \tau_2 - t_1)^2}{2}\left(1 + \frac{\mu w_2 - a_+}{\mu w_3 + a_+}\right) = \\ &= \frac{\mu(a_- - \mu w_1)(w_1 + w_2)\tau_1^2}{2(\mu w_2 + a_-)} + \frac{\mu(\mu w_2 - a_+)(w_2 + w_3)}{2(\mu w_3 + a_+)}(\tau_2 - q\tau_1)^2 \end{aligned} \tag{5.9}$$

Here we have introduced the notation

$$q = \frac{(\mu w_1 - a_-)}{(\mu w_2 + a_-)} \tag{5.10}$$

We transform the last expression in parentheses in equality (5.9) by substituting expressions (2.20) for τ_1 and τ_2 into it. We obtain

$$\tau_2 - q\tau_1 = \left(\frac{2L}{w_2}\right)^{\frac{1}{2}}\left[\left(\frac{w_1 + w_2}{w_1}\right)^{\frac{1}{2}}\frac{a_-}{\mu w_2 + a_-} + \left(\frac{w_3}{w_2 + w_3}\right)^{\frac{1}{2}}\right] \tag{5.11}$$

We now substitute expression (5.11), as well as expression (2.20) for τ_1, into (5.9). After some reduction, we have

$$x(T) = \mu L\left\{\frac{w_2(a_- - \mu w_1)}{w_1(\mu w_2 + a_-)} + \frac{(w_1 + w_2)(w_2 + w_3)(\mu w_2 - a_+)a_-^2}{w_1 w_2(\mu w_2 + a_-)^2(\mu w_3 + a_+)} + \right.$$
$$\left. + \frac{w_3(\mu w_2 - a_+)}{w_2(\mu w_3 + a_+)} + \frac{2(\mu w_2 - a_+)a_-}{w_2(\mu w_2 + a_-)(\mu w_3 + a_+)}\left[\frac{w_3(w_1 + w_2)(w_2 + w_3)}{w_1}\right]^{\frac{1}{2}}\right\} \tag{5.12}$$

We introduce the notation

$$r_1 = \left(1 + \frac{w_2}{w_1}\right)^{\frac{1}{2}} \tag{5.13}$$

and rewrite (5.12), expressing w_1 in terms of r_1 using equality (5.13). We obtain

$$x(T) = A_1 r_1^2 + A_2 r_1 + A_3 \tag{5.14}$$

$$A_1 = \frac{\mu L a_-}{\mu w_2 + a_-}\left[1 + \frac{(w_2 + w_3)(\mu w_2 - a_+)a_-}{w_2(\mu w_2 + a_-)(\mu w_3 + a_+)}\right]$$

$$A_2 = \frac{2\mu L(\mu w_2 - a_+)a_-[w_3(w_2 + w_3)]^{1/2}}{w_2(\mu w_2 + a_-)(\mu w_3 + a_+)}$$

$$A_3 = \mu L\left[-1 + \frac{w_3(\mu w_2 - a_+)}{w_2(\mu w_3 + a_+)}\right] = -\frac{\mu L(w_2 + w_3)a_+}{w_2(\mu w_3 + a_+)} < 0$$

Using equality (5.14), formula (2.21) for T, and the notation (5.13), we calculate the mean velocity of body M in mode a

$$V = \frac{x(T)}{T} = \frac{(A_1 r_1^2 + A_2 r_1 + A_3)}{\left(\frac{2L}{w_2}\right)^{\frac{1}{2}}(r_1 + r_3)}, \quad r_3 = \left(1 + \frac{w_2}{w_3}\right)^{\frac{1}{2}} \tag{5.15}$$

Let us investigate the dependence of the velocity V on the parameter $w_1 > 0$ for fixed values of w_2 and w_3 in mode a. The velocity V depends on w_1 only through r_1. We calculate the derivative of the velocity (5.15) with respect to r_1

$$\frac{\partial V}{\partial r_1} = \frac{A_1 r_1^2 + 2A_1 r_1 r_3 + A_2 r_3 - A_3}{\left(\frac{2L}{w_2}\right)^{\frac{1}{2}}(r_1 + r_3)^2} \tag{5.16}$$

When condition (5.5) is taken into account, the coefficients A_1, A_2 and A_3 satisfy the inequalities

$$A_1 > 0, \quad A_2 > 0, \quad A_3 < 0$$

Therefore, $\frac{\partial V}{\partial r_1} > 0$. Since the parameter r_1 increases monotonically as w_1 decreases by virtue of (5.13), the velocity V increases monotonically as w_1 decreases.

The values of w_1 in mode a have a lower limit defined by inequality (5.2). For some values of w_2 and w_3 that satisfy conditions (5.5) and (5.8), let the value $w_1 = \frac{a_-}{\mu}$ be allowed by conditions (5.4) and (5.7). Then the largest value of the velocity V is obtained in mode a in the limit for $w_1 = \frac{a_-}{\mu} + 0$, i.e., on the boundary between modes a and b. Note that mode b, in which there is no reverse motion of body M, is also more natural, simple and convenient for practical realization. Therefore, we will henceforth consider only mode b.

Integrating Eq. (1.6) for mode b with the acceleration $w(t)$ specified by the relations (2.11), we obtain

$$\begin{aligned} &\upsilon(t) = 0 \quad \text{for} \quad t \in [0, \tau_1] \\ &\upsilon(t) = (\mu w_2 - a_+)(t - \tau_1) \quad \text{for} \quad t \in [\tau_1, \tau_1 + \tau_2] \\ &\upsilon(t) = (\mu w_2 - a_+)\tau_2 - (\mu w_3 + a_+)(t - \tau_1 - \tau_2) \quad \text{for} \quad t \in [\tau_1 + \tau_2, t_2] \\ &\upsilon(t) = 0 \quad \text{for} \quad t \in [t_2, T] \end{aligned} \tag{5.17}$$

We will present the conditions that ensure mode b. For body M to remain at rest in the interval $[0, \tau_1]$, it is necessary and sufficient that

$$\mu w_1 \le a_- \tag{5.18}$$

As in the case of mode a, inequality (5.5) must hold. The time t_2 is determined from the condition $\upsilon(t_2) = 0$. By relations (5.17), we have

$$t_2 = \tau_1 + \tau_2 + \frac{\mu w_2 - a_+}{\mu w_3 + a_+}\tau_2 \tag{5.19}$$

The condition $t_2 \le T$ leads to the inequality

$$(\mu w_2 - a_+)\tau_2 \le (\mu w_3 + a_+)\tau_3 \tag{5.20}$$

If $t_2 < T$, the state of rest of body M is maintained in the interval $[t_2, T]$. Therefore, a condition similar to (5.18) should hold for the strict inequality in (5.20). We have

$$\mu w_3 \le a_- \quad \text{When} \quad (\mu w_2 - a_+)\tau_2 < (\mu w_3 + a_+)\tau_3 \tag{5.21}$$

If there is an equality sign in condition (5.20), i.e., if

$$(\mu w_2 - a_+)\tau_2 = (\mu w_3 + a_+)\tau_3 \tag{5.22}$$

satisfaction of the condition $\mu w_3 \le a_-$ is not required. We shall henceforth keep in mind that one of two constraints may hold: either the combination of inequalities (5.21) or the alternative condition defined by equality (5.22).

We calculate the displacement $x(T)$ as the area of the triangle under the graph of the function $\upsilon(t)$ in Fig. 7b:

$$x(T) = \frac{\upsilon(\tau_1 + \tau_2)(t_2 - \tau_1)}{2} \tag{5.23}$$

Substituting the expression for $\upsilon(t)$ (5.17) and the expression for t_2 (5.19) into (5.23), we obtain

$$x(T) = \frac{\mu(w_2 + w_3)(\mu w_2 - a_+)\tau_2^2}{2(\mu w_3 + a_+)} \tag{5.24}$$

We calculate the mean velocity using equalities (1.11) and (5.24) and formulae (2.20) and (2.21) for τ_2 and T. Here we again employ the notation r_1 and r_3 as defined in (5.13) and (5.15). As a result, for mode b we obtain

$$V = \frac{x(T)}{T} = \frac{\mu\left(\frac{L}{2}\right)^{\frac{1}{2}}(w_2 + w_3)(\mu w_2 - a_+)(r_1 + r_3)}{w_2^{\frac{1}{2}}(\mu w_3 + a_+)r_1^2 r_3^2} \tag{5.25}$$

In the same way as with formulae (3.14), we change to the dimensionless variables

$$w_i = \frac{a_- y_i}{\mu}, \quad i = 1, 2, 3, \quad V = \left(\frac{\mu L a_-}{2}\right)F, \quad c = \frac{a_+}{a_-} \tag{5.26}$$

Then, taking into account equalities (5.13) and (5.15) for r_1 and r_3, we reduce (5.25) to the form

$$F(y) = \frac{(y_2 - c)\left[y_1^{\frac{1}{2}} y_3 (y_1 + y_2)^{\frac{1}{2}} + y_1 y_3^{\frac{1}{2}} (y_2 + y_3)^{\frac{1}{2}}\right]}{y_2^{\frac{1}{2}} (y_1 + y_2)(y_3 + c)} \tag{5.27}$$

Here $y = (y_1, y_2, y_3)$ is a three-dimensional vector.

In the dimensionless parameters (5.26) the constraints (5.18) and (5.5) take the form

$$0 < y_1 \le 1, \quad y_2 > c \tag{5.28}$$

To reduce conditions (5.21) and (5.22) to dimensionless form, we substitute expressions (5.26) into formulae (2.20) and calculate the ratio

$$\tau^0 = \frac{\tau_2}{\tau_3} = \frac{y_3 (y_1 + y_2)^{\frac{1}{2}} + y_1^{\frac{1}{2}} y_3^{\frac{1}{2}} (y_2 + y_3)^{\frac{1}{2}}}{y_2 (y_1 + y_2)^{\frac{1}{2}}} \tag{5.29}$$

We substitute the ratio (5.29) into conditions (5.21) and (5.22) and change to the dimensionless variables (5.26) under these conditions. When the second inequality in (5.28) is taken into account, condition (5.21) becomes

$$0 < y_3 \le 1 \quad \text{when} \quad \tau^0 < \frac{(y_3 + c)}{(y_2 - c)} \tag{5.30}$$

We express the second inequality in (5.30) explicitly in terms of y_i ($i = 1, 2, 3$) using (5.29). Introducing the notation

$$\psi(y) = c^2 (y_1 + y_2) + y_3 (2 y_1 c + c^2 - y_1 y_2) \tag{5.31}$$

we represent condition (5.30) in the form

$$0 < y_3 \le 1 \quad \text{for} \quad \psi(y) > 0 \tag{5.32}$$

The alternative condition (5.22) takes the form

$$\psi(y) = 0 \tag{5.33}$$

Mode b is feasible under the condition $y \in D$. The region D is specified by inequalities (5.28), as well as by conditions (5.32) and (5.33).

6. Optimization of the three-phase motion

Let us find the values of the parameters $w_i > 0$ ($i = 1, 2, 3$) for which the highest mean velocity V of body M is reached in three-phase mode b. This problem reduces to determining the vector y that gives a maximum of the function F (5.27). As was done in Section 4, initially we do not impose the upper limits (2.22) on the parameters sought, i.e., we first seek the maximum of F for $y \in D$, and we then introduce these constraints.

Calculating the partial derivatives of the function F (5.27) with respect to y_i ($i = 1, 2, 3$), we obtain the inequality

$$\frac{\partial F}{\partial y_1} > 0 \tag{6.1}$$

for all $y_i > 0$ ($i = 1, 2, 3$) and the relations

$$\frac{\partial F}{\partial y_2} = G_1 + G_2 (2 y_1 + 2c - y_3) \tag{6.2}$$

$$\frac{\partial F}{\partial y_3} = G_3\left[2c(y_1+y_2)^{\frac{1}{2}}(y_2+y_3)^{\frac{1}{2}}y_3 + y_1^{\frac{1}{2}}(2cy_3 - y_2y_3 + cy_2)\right] \tag{6.3}$$

Here G_j ($j = 1, 2, 3$) are functions of y that are positive for all $y_i > 0$ ($i = 1, 2, 3$).

6.1. The case without the constraints (2.22)

It follows from inequality (6.1) that the function F does not have inner extrema in the region D and that the maximum is reached on the boundary of this region, which possibly includes points at infinity.

According to (5.28), (5.32) and (5.33), the boundary of the region D consists of portions of the planes $y_1 = 0$, $y_1 = 1$, $y_2 = c$, $y_3 = 0$ and $y_3 = 1$ and of the surface $\psi(y) = 0$. When any of the conditions $y_1 = 0$, $y_2 = c$, $y_3 = 0$ holds, the function F (5.27) vanishes everywhere, and the maximum required is not reached on the respective portions of the boundary.

On the portion of the boundary $y_3 = 1$, the maximum can be reached by virtue of condition (6.1) only on one of the boundaries of this portion, i.e., at $y_1 = 1$ or $\psi(y) = 0$. Therefore, it is sufficient to consider only the portions of the boundary of the region D where $y_1 = 1$ or $\psi(y) = 0$.

6.1.1. Let first $y_1 = 1$ and $\psi(y) \neq 0$

Then condition (5.32) must hold. When $y_1 = 1$ and equality (5.31) is taken into account, this condition splits into two conditions:

$$\begin{aligned} &0 < y_3 \le 1 \quad \text{for} \quad y_2 \le c(c+2) \\ &0 < y_3 \le 1 \quad \text{and} \quad y_3 < \psi_1(y_2) \quad \text{for} \quad y_2 > c(c+2) \end{aligned} \tag{6.4}$$

Here we have introduced the notation

$$\psi_1(y_2) = \frac{c^2(y_2+1)}{y_2 - c(c+2)} \quad \text{for} \quad y_2 > c(c+2) \tag{6.5}$$

Since in the case under consideration, $y_1 = 1$ and, according to conditions (6.4), $y_3 \le 1$, it follows from (6.2) that $\frac{\partial F}{\partial y_2} > 0$. Therefore, in this case the maximum of the function F with respect to y_2 is reached as $y_2 \to \infty$. Substituting $y_1 = 1$ and $y_2 = \infty$ into equality (5.27), we obtain

$$F = y_3^{\frac{1}{2}}\left(y_3^{\frac{1}{2}} + 1\right)(y_3 + c)^{-1} \tag{6.6}$$

According to (6.4) and (6.5), as $y_2 \to \infty$, the range of variation of y_3 becomes the interval

$$\begin{aligned} &0 < y_3 < c^2 \quad \text{for} \quad c < 1 \\ &0 < y_3 \le 1 \quad \text{for} \quad c \ge 1 \end{aligned} \tag{6.7}$$

As a simple investigation shows, the function (6.6) has a single maximum with respect to y_3 when $y_3 \ge 0$, which is reached when

$$y_3 = y_3^0 = c + 2c^2 + 2c(c + c^2)^{\frac{1}{2}}$$

It is not difficult to verify that the inequality $y_3^0 > \min(1, c^2)$ holds for both $c < 1$ and $c \ge 1$, so that the value y_3^0 lies outside the interval defined by (6.7) for both $c < 1$ and $c \ge 1$. Therefore, the function (6.6) increases strictly in the interval (6.7) and reaches its maximum on the right-hand boundary of this interval. We have

$$\begin{aligned} &\max F = 1, \quad y_2 = \infty, \quad y_3 = c^2 - 0 \quad \text{for} \quad c < 1 \\ &\max F = 2(c+1)^{-1}, \quad y_2 = \infty, \quad y_3 = 1 \quad \text{for} \quad c \ge 1 \end{aligned} \tag{6.8}$$

Result (6.8) was obtained under the conditions $y_1 = 1$ and $\psi(y) \neq 0$.

6.1.2. Let us consider the second possibility for reaching a maximum of the function F, namely, the surface $\psi(y) = 0$

Using equality (5.31), in this case we find

$$y_3 = \frac{c^2(y_1+y_2)}{y_1y_2 - 2y_1c - c^2} \quad \text{for} \quad y_1y_2 - 2y_1c - c^2 > 0 \tag{6.9}$$

We substitute this expression for y_3 into (5.27). After some simplification, we obtain

$$F = (y_1^{-1} + y_2^{-1})^{-\frac{1}{2}} \tag{6.10}$$

The maximum of the function (6.10) with respect to y_1, y_2 under the constraints (5.28) is reached for $y_1 = 1$, $y_2 = \infty$ and is equal to $F = 1$. From (6.9) we obtain $y_3 = c^2$ at this maximum for any c.

Comparing these results with (6.8), we reach the conclusion that the absolute maximum of the function F with respect to $y \in D$ is specified by the equalities

$$y_1 = 1, \quad y_2 = \infty, \quad y_3 = c^2, \quad F = 1 \tag{6.11}$$

for any $c > 0$.

We find the corresponding dimensional variables from formulae (5.26), (2.20) and (5.19) in the form

$$\begin{aligned} &w_1 = \frac{a_-}{\mu}, \quad w_2 \to \infty, \quad w_3 = \frac{a_+^2}{a_-\mu}, \quad V = \left(\frac{\mu L a_-}{2}\right)^{\frac{1}{2}} \\ &\tau_1 = \left(\frac{2\mu L}{a_-}\right)^{\frac{1}{2}}, \quad \tau_2 = 0, \quad \tau_3 = \frac{(2\mu L a_-)^{\frac{1}{2}}}{a_+}, \quad t_2 = \tau_1 + \tau_3 = T \end{aligned} \tag{6.12}$$

Note some specific features of the optimum mode (6.12) just constructed.

At $t = \tau_1$ there is a jump in the velocity of body M. Using formulae (5.17), (6.12) and (2.20) and evaluating the expression obtained, we find the magnitude of this jump

$$\Delta \upsilon = \upsilon(\tau_1 + 0) - \upsilon(\tau_1 - 0) = \mu w_2 \tau_2 = \mu(2L)^{\frac{1}{2}}\left(w_1^{\frac{1}{2}} + w_3^{\frac{1}{2}}\right) = \left(\frac{2\mu L}{a_-}\right)^{\frac{1}{2}}(a_+ + a_-) \tag{6.13}$$

According to (5.17) and (6.13), the velocity $\upsilon(t)$ of body M in the optimum mode (6.12) varies as follows:

$$\begin{aligned} &\upsilon(t) = 0 \quad \text{for} \quad t \in (0, \tau_1), \quad \upsilon(\tau_1 + 0) = \Delta\upsilon \\ &\upsilon(t) = \Delta\upsilon - \left(\frac{a_+}{a_-}\right)(a_+ + a_-)(t - \tau_1) \quad \text{for} \quad t \in (\tau_1, T) \\ &\upsilon(T) = 0 \end{aligned} \tag{6.14}$$

Here there is no rest interval of body M at the end of the period. A graph of the velocity $\upsilon(t)$ for the optimum mode is shown in Fig. 7c.

In the case of isotropic dry friction ($a_+ = a_- = a$), according to formulae (6.12) and (6.13), we have

$$\begin{aligned} &w_1 = w_3 = \frac{a}{\mu}, \quad w_2 = \infty, \quad \tau_1 = \tau_3 = \left(\frac{2\mu L}{a}\right)^{\frac{1}{2}}, \quad \tau_2 = 0 \\ &\Delta\upsilon = 2(2\mu L a)^{\frac{1}{2}}, \quad V = \left(\frac{\mu L a}{2}\right)^{\frac{1}{2}} \end{aligned} \tag{6.15}$$

6.2. The case with constraints (2.22)

We will now consider the problem of maximizing the mean velocity V when there are upper limits (2.22) for the relative acceleration. We will write these constraints in the dimensionless variables (5.26)

$$0 < y_i \le Y, \quad i = 1, 2, 3, \quad Y = \frac{\mu W}{a_-} \tag{6.16}$$

The optimization problem reduces to finding the maximum of the function F (5.27) over the region D^*, which is the intersection of the region D defined by conditions (5.28), (5.32) and (5.33) with the set specified by the constraints (6.16).

Combining conditions (5.28) and (6.16), we find that the region D^* is a non-empty set if

$$Y > c \tag{6.17}$$

and the parameters y_i satisfy the inequalities

$$0 < y_1 \le \min(1, Y), \quad c < y_2 \le Y, \quad 0 < y_3 \le Y \tag{6.18}$$

Taking conditions (5.32) and (5.33) into account, we find that $y \in D^*$ if either the conditions

$$0 < y_1 \le \min(1, Y), \quad c < y_2 \le Y, \quad 0 < y_3 \le \min(1, Y), \quad \psi(y) > 0 \tag{6.19}$$

or the conditions

$$0 < y_1 \le \min(1, Y), \quad c < y_2 \le Y, \quad 0 < y_3 \le Y, \quad \psi(y) = 0 \tag{6.20}$$

are satisfied simultaneously.

As in the case without constraints (2.22), by virtue of inequality (6.1), a maximum of F cannot be reached within the region D^* on the planes $y_1 = 0$, $y_2 = c$ and $y_3 = 0$, where we have $F = 0$ according to (5.27). Furthermore, by virtue of inequality (6.1), on the segments of the planes $y_3 = 1$, $y_3 = Y$ and $y_2 = Y$, a maximum of F can only be reached on the boundaries of these segments, i.e., when one more of the remaining constraints (6.19) and (6.20) holds, i.e., either when $y_1 = \min(1, Y)$ or when $\psi(y) = 0$.

As a result, conditions (6.19) together with inequality (6.17), reduce to the following two cases:

$$y_1 = 1, \quad c < y_2 \le Y, \quad 0 < y_3 \le 1, \quad \psi(y) > 0 \text{ for } Y > \max(1, c) \tag{6.21}$$

$$y_1 = Y, \quad c < y_2 \le Y, \quad 0 < y_3 \le Y, \quad \psi(y) > 0 \text{ for } c < Y \le 1 \tag{6.22}$$

A third possible case is specified by the conditions (6.20) and (6.17).

We will examine these three cases in succession.

6.2.1. Let conditions (6.21) hold

From (6.2), for $y_1 = 1$, $y_3 \le 1$ we have, as in case 6.1.1

$$\frac{\partial F}{\partial y_2} > G_2(2 + 2c - 2y_3) > 0$$

Therefore, under the conditions (6.21) a maximum of F is reached at $y_2 = Y$. When the relations $y_1 = 1$, $y_2 = Y$ and (5.31) are taken into account, the inequality $\psi(y) > 0$ from (6.21) becomes

$$c^2(1 + Y) + y_3(2c + c^2 - Y) > 0 \tag{6.23}$$

and equality (6.3) takes the form

$$\frac{\partial F}{\partial y_3} = G_3\left[2c(1 + Y)^{\frac{1}{2}}(Y + y_3)^{\frac{1}{2}}y_3^{\frac{1}{2}} + 2cy_3 - Yy_3 + cY\right] \tag{6.24}$$

Here $G_3 > 0$.

Let us find the lower estimate of the derivative (6.24) for $Y>0$, $0<y_3\le 1$. We have

$$\frac{\partial F}{\partial y_3} > G_3(4cy_3 - Yy_3 + cY) \tag{6.25}$$

Hence

$$\frac{\partial F}{\partial y_3} > G_3[4y_3 + Y(1-y_3)] > 0 \quad \text{for } c \ge 1$$

$$\frac{\partial F}{\partial y_3} > G_3 cY > 0 \text{ for } c<1, \quad Y \le 4c$$

In the case when $c<1$, $Y>4c$, we rewrite estimate (6.25) in the form

$$\frac{\partial F}{\partial y_3} > G_3(Y-4c)[\psi_2(Y) - y_3] \tag{6.26}$$

where we have introduced the notation

$$\psi_2(Y) = cY(Y-4c)^{-1} \tag{6.27}$$

Note that the function $\psi_2(Y)$ decreases monotonically as Y increases.

We will separately examine the cases

$$Y \in (4c, 4c(1-c)^{-1}), \quad Y \ge 4c(1-c)^{-1} \tag{6.28}$$

which may be encountered when $c<1$ and $Y>4c$. When the notation (6.27) is taken into account in the first case in (6.28), we obtain

$$\psi_2(Y) > \psi_2(4c(1-c)^{-1}) = 1 \tag{6.29}$$

As a consequence of (6.29), the estimate (6.26) yields $\frac{\partial F}{\partial y_3} > 0$ in the first case in (6.28), since $y_3 \le 1$.

In the second case in (6.28), the inequality $Y > c(c+2)$ obviously holds. Therefore, condition (6.23) can be represented in the form $y_3 < \psi_1(Y)$, where the function ψ_1 is defined by formula (6.5). Consequently, the estimate (6.26) takes the form

$$\frac{\partial F}{\partial y_3} > G_3(Y-4c)[\psi_2(Y) - \psi_1(Y)]$$

Substituting the functions $\psi_1(Y)$ and $\psi_2(Y)$ from formulae (6.5) and (6.27), respectively, into this inequality, we obtain

$$\frac{\partial F}{\partial y_3} > \frac{G_3 c[(1-c)Y^2 - 3c(1-c)Y + 4c^2]}{Y - c(c+2)} \tag{6.30}$$

In the second case in (6.28), we have

$$c<1, \quad Y \ge 4c(1-c)^{-1} > 3c$$

and the inequality $\frac{\partial F}{\partial y_3} > 0$ follows from the estimate (6.30).

Thus, it has been proved that $\frac{\partial F}{\partial y_3} > 0$ under the conditions $y_2 = Y$ and (6.21).

Therefore, in case 6.2.1 a maximum of the function F is obtained for $y_1 = 1$, $y_2 = Y$ and the largest value $y_3 = 1$ allowed by conditions (6.21). The inequality $\psi(y) > 0$ from (6.21) must still be verified. When $y_1 = 1$ and $y_2 = Y$, this inequality becomes (6.23). Substituting $y_3 = 1$ into it, after some simplification we obtain the condition

$$2c + (c-1)Y > 0 \tag{6.31}$$

We introduce the notation

$$Y_0(c) = 2c/(1-c), \quad c<1 \tag{6.32}$$

Condition (6.31) holds for $c \geq 1$, and the condition $Y < Y_0(c)$ holds for $c < 1$.

As a result, we find that the required maximum for the function F (5.27) in case 6.2.1 is specified by the relations

$$y_1 = 1, \quad y_2 = Y, \quad y_3 = 1, \quad F = \frac{2(Y-c)}{(1+c)Y^{\frac{1}{2}}(Y+1)^{\frac{1}{2}}}$$
$$\text{for} \quad c \geq 1, \quad Y > c \quad \text{and for} \quad c<1, \quad 1 < Y < Y_0(c) \tag{6.33}$$

6.2.2. Let conditions (6.22) hold

When $y_1 = Y$ and $y_3 \leq Y$, from relation (6.2) we have

$$\frac{\partial F}{\partial y_2} > G_2(2Y + 2c - y_3) > 0$$

Hence, as in case 6.1.1, we obtain $y_2 = Y$. The inequality $\psi(y) > 0$ again takes the form (6.23), and relation (6.3) can be reduced to the form (6.24). The proof of the inequality $\frac{\partial F}{\partial y_3} > 0$ given in case 6.2.1 was verified for any $Y > 0$ and $0 < y_3 \leq 1$, and, therefore, remains valid in the case under consideration.

Consequently, a maximum of the function F is reached in this case for $y_2 = Y$ and the largest value $y_3 = Y$ allowed by conditions (6.22).

To verify the condition $\psi(y) > 0$, we substitute $y_3 = Y$ into inequality (6.23). After some reduction, we obtain

$$Y^2 - 2c(c+1)Y - c^2 < 0 \tag{6.34}$$

The single positive root of the quadratic trinomial in (6.34) equals

$$Y^*(c) = c\left[c + 1 + (c^2 + 2c + 2)^{\frac{1}{2}}\right] \tag{6.35}$$

The quadratic trinomial in (6.34) is negative for $Y < Y^*(c)$ and positive for $Y > Y^*(c)$. Therefore, inequality (6.34) holds for $Y < Y^*(c)$. The inequalities $c < Y \leq 1$ from (6.22) should also hold in case 6.2.2. Thus, we have the feasibility condition of case 6.2.2 in the form

$$c < Y < \min(1, Y^*(c)) \tag{6.36}$$

Note that, by virtue of equality (6.35), we have $Y^*(c) > c$ for all $c > 0$, and the condition $1 < Y^*(c)$ can be reduced to the form

$$c^2 + 2c(c+1) - 1 > 0$$

This inequality, in turn, can be reduced to the form $c > \frac{1}{3}$. Thus, we have

$$Y^*(c) \leq 1 \quad \text{for} \quad c \leq \frac{1}{3}; \quad Y^*(c) > 1 \quad \text{for} \quad c > \frac{1}{3} \tag{6.37}$$

Substituting $y_1 = y_2 = y_3 = Y$ into (5.27) and taking into account inequalities (6.36) and (6.37), we obtain the required solution in case 6.2.2

$$y_1 = y_2 = y_3 = Y, \quad F = (2Y)^{\frac{1}{2}}\frac{Y-c}{Y+c}$$
$$\text{for} \quad c \leq \frac{1}{3}, \quad c < Y \leq Y^*(c) \quad \text{and for} \quad c > \frac{1}{3}, \quad c < Y < 1 \tag{6.38}$$

6.2.3. Let us assume that conditions (6.20) hold

From the equality $\psi(y)=0$, we express y_3 in the form (6.9), as in case 6.1.2. Substituting expression (6.9) into equality (5.27), we obtain F in the form of (6.10). Since the function F (6.10) increases monotonically with respect to y_1 and y_2, the required maximum is reached at the largest values of y_1 and y_2 that are allowed by conditions (6.20). The combination of these constraints is given by the inequalities

$$0 < y_1 \le \min(1, Y), \quad c < y_2 \le Y \tag{6.39}$$

as well as by the condition $y_3 \le Y$, which can be transformed using equality (6.9) into the inequality

$$y_2(Yy_1 - c^2) \ge (2Yc + c^2)y_1 + c^2 Y \tag{6.40}$$

In the y_1, y_2 plane inequalities (6.39) specify a rectangle, and inequality (6.40) specifies a hyperbola with asymptotes $y_1 = c^2Y^{-1}$ and $y_2 = 2c + c^2Y^{-1}$, which is represented by curve H in Fig. 8. The points allowed by conditions (6.39) and (6.40) lie to the right of and above hyperbola H and, at the same time, within the rectangle (6.39). Since the function F (6.10) increases monotonically with respect to both y_1 and y_2, the required maximum always lies at point A with coordinates (see Fig. 8)

$$y_1 = \min(1, Y), \quad y_2 = Y \tag{6.41}$$

only if the values (6.41) satisfy inequality (6.40). Otherwise, this version of mode b is not feasible. The value of F is given by formula (6.10), and the value of y_3 is given by (6.9).

Let us consider the cases of $Y \ge 1$ and $Y < 1$ separately.

When $Y \ge 1$, from (6.40) and (6.41) we have

$$y_1 = 1, \quad y_2 = Y, \quad Y^2 - 2c(c+1) - c^2 \ge 0$$

When the results of the analysis of the quadratic trinomial in (6.34) are taken into account, the last inequality can be represented in the form $Y \ge Y^*(c)$, where $Y^*(c)$ is specified by (6.35). Taking into account equalities (6.9) and (6.10), we obtain

$$y_1 = 1, \quad y_2 = Y, \quad y_3 = c^2\frac{1+Y}{Y-2c-c^2}, \quad F = \left[\frac{Y}{Y+1}\right]^{\frac{1}{2}} \quad \text{for} \quad Y \ge \max(1, Y^*(c)) \tag{6.42}$$

For $Y < 1$, from (6.40) and (6.41) we obtain

$$y_1 = y_2 = Y, \quad Y^2 - 2Yc - 3c^2 = (Y+c)(Y-3c) \ge 0$$

Using formulae (6.9) and (6.10), for the case under consideration we find

$$y_1 = y_2 = Y, \quad y_3 = \frac{2c^2Y}{Y^2 - 2cY - c^2}, \quad F = \left(\frac{Y}{2}\right)^{\frac{1}{2}} \quad \text{for} \quad 3c \le Y < 1 \tag{6.43}$$

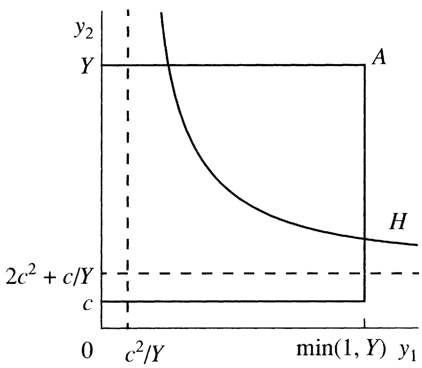

Fig. 8.

After completing the analysis of all the possibilities, we can bring together the results obtained for case 6.2. These results and their regions of existence are given by formulae (6.33) for case 6.2.1, by (6.38) for case 6.2.2 and by (6.42) and (6.43) for case 6.2.3. The solutions were obtained in the region $Y > c$ of the (c, Y) parameter plane.

However, as can be observed from the solutions presented, their existence regions in the (c, Y) plane do not fully match one another. There are two zones in the region $Y > c$ that require additional consideration.

Direct checking can assure us of the validity of the inequalities

$$Y^*(c) < 3c \quad \text{for } c \in \left(0, \frac{1}{3}\right), \quad Y^*(c) < Y_0(c) \quad \text{for } c \in \left(\frac{1}{3}, 1\right) \tag{6.44}$$

The functions $Y^*(c)$ and $Y_0(c)$ are given by formulae (6.35) and (6.32), respectively.

It follows from the first inequality in (6.44) that the (c, Y) plane contains a non-empty region S_1 defined by the conditions

$$Y^*(c) < Y < 3c, \quad c \in (0, 1/3) \tag{6.45}$$

The solution was not determined in the region (6.45). The solutions (6.38) and (6.43), which are adjacent to the region S_1, exist in the regions $Y \le Y^*(c)$ and $Y \ge 3c$, respectively.

Thus, mode b is impossible in the region S_1.

The second inequality in (6.44) specifies the region S_2 by the conditions

$$Y^*(c) < Y < Y_0(c), \quad c \in \left(\frac{1}{3}, 1\right) \tag{6.46}$$

Both solution (6.33) and solution (6.42) exist in the region (6.46). Comparing the corresponding maximum values of F given by formulae (6.33) and (6.42), we conclude that the value of F from (6.42) is greater than the value from (6.33) in the region (6.46). Therefore, in the region S_2 we should choose solution (6.42), and the curve $Y = Y^*(c)$, on which the function F has a jump, will be the boundary of the regions on which F is defined by (6.33) and (6.42).

On the basis of formulae (6.33), (6.38), (6.42) and (6.43) and an analysis of the regions (6.45) and (6.46), we obtain the following results

$$\begin{aligned}
&1)\ y_1 = 1, \quad y_2 = Y, \quad y_3 = 1, \quad F = \frac{2(Y-c)}{(1+c)Y^{\frac{1}{2}}(Y+1)^{\frac{1}{2}}} \\
&\quad \text{for} \quad c \ge 1, \quad Y > c \ \text{ and for } \ \frac{1}{3} < c < 1, \quad 1 < Y < Y^*(c) \\
&2)\ y_1 = y_2 = y_3 = Y, \quad F = (2Y)^{\frac{1}{2}}\frac{Y-c}{Y+c} \\
&\quad \text{for} \quad c \le \frac{1}{3}, \quad c < Y \le Y^*(c) \ \text{ and for } \ c > \frac{1}{3}, \quad c < Y < 1 \\
&3)\ y_1 = 1, \quad y_2 = Y, \quad y_3 = c^2\frac{Y+1}{Y-2c-c^2}, \quad F = \left[\frac{Y}{Y+1}\right]^{\frac{1}{2}} \ \text{ for } Y \ge 1, \quad Y \ge Y^*(c) \\
&4)\ y_1 = Y, \quad y_2 = Y, \quad y_3 = \frac{2c^2Y}{Y^2-2cY-c^2}, \quad F = \left(\frac{Y}{2}\right)^{\frac{1}{2}} \ \text{ for } 3c \le Y < 1
\end{aligned} \tag{6.47}$$

Figs. 9 and 10 show the straight lines $c = 1$, $Y = 1$ and $Y = c$, the segments of the straight lines $c = \frac{1}{3}$ and $Y = 3c$ for $Y \le 1$, the curve $Y = Y^*(c)$ and the segment of the curve $Y = Y_0(c)$ for $c \ge \frac{1}{3}$, which is depicted by dashed lines, in the (c, Y) plane on different scales. The numbers in Figs. 9 and 10 correspond to the numbers of the cases in (6.47).

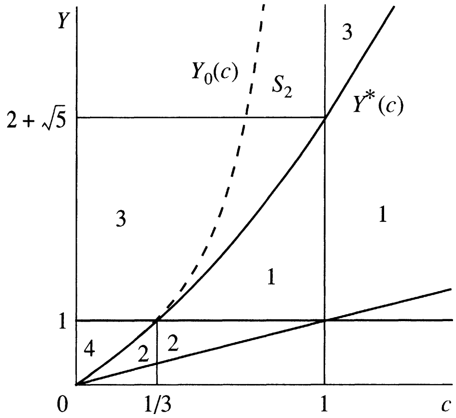

Fig. 9.

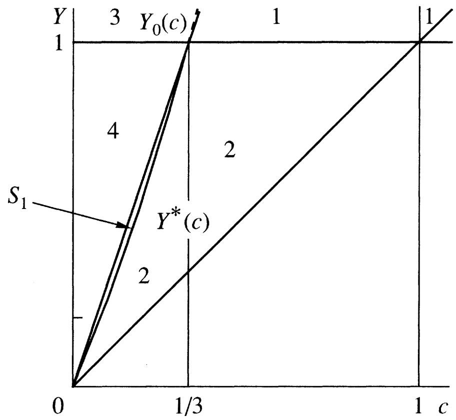

Fig. 10.

In the case of isotropic dry friction ($c = 1$), relations (6.47) take the form

$$
\begin{aligned}
&y_1 = 1, \quad y_2 = Y, \quad y_3 = 1, \quad F = \frac{(Y-1)}{Y^{\frac{1}{2}}(Y+1)^{\frac{1}{2}}} \quad \text{for } 1 < Y \le 2+\sqrt{5} \\
&y_1 = 1, \quad y_2 = Y, \quad y_3 = \frac{Y+1}{Y-3}, \quad F = \left(\frac{Y}{Y+1}\right)^{\frac{1}{2}} \quad \text{for } Y > 2+\sqrt{5}
\end{aligned}
\tag{6.48}
$$

The transition to the original dimensional variables in solutions (6.47) and (6.48) can be accomplished using formulae (5.26), (6.16) and (2.20), (2.21).

7. Discussion of the results

Thus, the optimum two- and three-phase motions in the presence of anisotropic dry friction have been fully constructed both when there are no constraints (2.10) on the relative velocity (2.22) and constraints (2.22) on the relative acceleration of mass m, and when these constraints are present.

We note several characteristic features of the optimum motions obtained. When there are no constraints (2.10), the mean velocity for the two-phase motion lies in the range (4.11), i.e.,

$$V \sim \frac{(\mu L a_-)^{\frac{1}{2}}}{2}$$

The mean velocity for the three-phase motion when there are no constraints (2.22) is specified by the fourth relation in (6.12)

$$V = \left(\frac{\mu L a_-}{2}\right)^{\frac{1}{2}}$$

In both cases the mean velocities of body M are bounded for any permissible (including arbitrarily large) relative velocities and accelerations of mass m. The maximum velocities for the cases of the two-phase and three-phase motions differ only in the coefficients: the velocity is greater for the three-phase motion.

Furthermore, despite the significant differences in the starting assumptions for the two- and three-phase modes, there is remarkable similarity between the corresponding optimum modes. Comparing Figs. 4c and 7c, we see that the three-phase optimum mode degenerates into the two-phase mode, in which the relative velocity of mass m undergoes a jump between phases and is not constant within the phases.

When constraints (2.10) are imposed on the velocity of mass m, the two-phase motion is possible only if the dimensionless variable X is bounded from below by (4.15). Similarly, when constraints (2.22) are imposed on the acceleration, the three-phase motion is possible only if the dimensionless variable Y is bounded from below by (6.17).

As the dimensionless velocity X and the dimensionless acceleration Y approach infinity, the optimum solutions obtained taking of the constraints into account transform, as expected, into the optimum solutions when there are no constraints. More specifically, solutions (4.13) and (4.14) transform into the corresponding solutions (4.5) and (4.6) as $X \to \infty$, and solution (6.47) transforms into solution (6.12) as $Y \to \infty$.

The locomotion principle investigated in this paper is of interest in relation to the development of some types of mobile robots and is applicable, in particular, for robots that move inside pipes.

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